Chapter I: Necessary Elements of Calculus

Usually a course in mechanics has some calculus as a prerequisite. Nonetheless, I will assume that you know nothing about derivatives or integrals and will try to supply you with these necessary tools of calculus in this chapter. While it may be true that this, as well as the other sections on mathematics, will be repulsive to a real mathematician it should be sufficient for our purpose. You will have plenty of practice with these abstract ideas in the chapter immediately following, but make sure you can work the problems at the end of this chapter before going on.

Introduction to Derivatives

Suppose you have a function of x which we will denote by $f(x)$. For example, you could have $f(x) = kx^2$ where k is some constant. For any given function you could evaluate the function at any values of x. In particular you could find, for two points x_1 and x_2, the numbers $f(x_1)$ and $f(x_2)$. You could then calculate

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Suppose we let x_1 be any arbitrary value, x, and we take x_2 to be just a bit larger than this, $x_2 = x + \Delta x$. Then the above ratio becomes

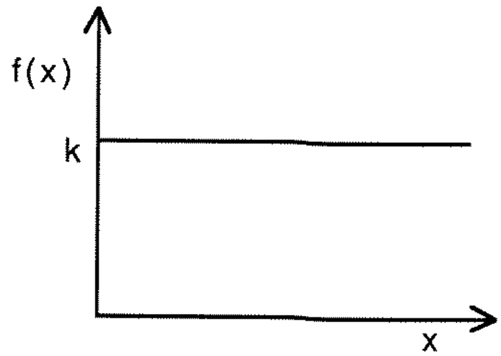
$$\frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

We call this $\frac{\Delta f}{\Delta x}$ and, as you see, it measures the change in the value of the function when the variable is changed by an amount Δx. For most functions $\frac{\Delta f}{\Delta x}$ will be different for different values of Δx. It turns out to be very convenient to have something like $\frac{\Delta f}{\Delta x}$ which does not depend on the size of Δx. For the kind of functions one runs into in this course if you make Δx smaller and smaller the value of $\frac{\Delta f}{\Delta x}$ ultimately stops changing. Making Δx smaller and smaller is mathematically called "taking the limit as Δx approaches zero." The value of $\frac{\Delta f}{\Delta x}$ is, in this case, called the "derivative of f with respect to x" and is given the symbol $\frac{df}{dx}$. To summarize

$$\frac{df}{dx} = limit_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Before concluding that this messy looking thing is very complicated, let's consider a few examples.

Consider the simplest of all functions $f(x) = k$ where k is a constant. If we draw a graph of this function it looks like



No matter what value x has, the function is always equal to k. Then

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{k-k}{\Delta x} = 0$$

no matter how small we make Δx. Therefore, if

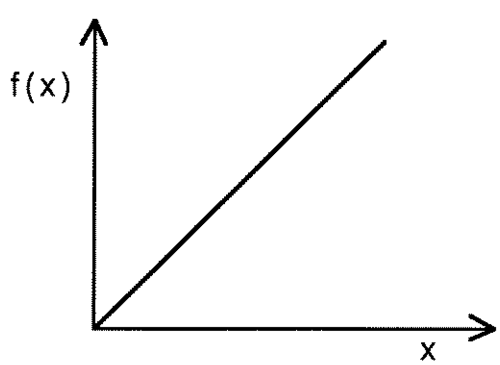
$$f(x) = k$$

then

$$\frac{df}{dx} = 0.$$

In this example we never needed to "take the limit as Δx approached zero" because $\frac{\Delta f}{\Delta x}$ didn't depend on the size of Δx.

Consider now the next most simple function, $f(x) = x$. Such a function is shown below



Now if we calculate $\frac{\Delta f}{\Delta x}$, starting at any x, we find

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{(x+\Delta x) - x}{\Delta x} = \frac{\Delta x}{\Delta x} = 1$$

and again $\frac{\Delta f}{\Delta x}$ is independent of the size of Δx. Thus if we let Δx get very small we still have the same value. Therefore, if

$$f(x) = x$$

then

$$\frac{df}{dx} = 1$$

and again all the bother with the limit business was irrelevant.

Exercise. Show that if $f(x) = kx$, then $\frac{df}{dx} = k$.

We do one last example, the function $f(x) = kx^2$ where k is a constant. This is, in fact, the most complicated function we will have to deal with for most of the course. Again we start by calculating $\frac{\Delta f}{\Delta x}$ for any value of Δx starting at any x:

$$\begin{aligned}\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{k(x+\Delta x)^2 - kx^2}{\Delta x} \\ &= \frac{kx^2 + 2kx\Delta x + k(\Delta x)^2 - kx^2}{\Delta x} \\ &= 2kx + k\Delta x.\end{aligned}$$

Now it is clear that the value of $\frac{\Delta f}{\Delta x}$ depends on how small we make Δx. If we let Δx go zero then the term $k\Delta x$ also goes to zero. We have, then, if

$$f(x) = kx^2$$

then

$$\frac{df}{dx} = 2kx.$$

One can use the procedure above to find the derivative of any function. If we tabulate the results we have just obtained, including those from the exercise,

$f(x)$	$\frac{df}{dx}$
k	0
kx	k
kx^2	$2kx$

we might see a pattern. In fact you should convince yourself that the general formula

$$\frac{d(kx^n)}{dx} = nkx^{n-1} \quad (n \;\; any \;\; integer)$$

gives back the results we have obtained for n = 0, 1 and 2 and we will assume (its even correct) that it works for any number. The process of obtaining a derivative is called *differentiating a function with respect to its variable* and we can now differentiate any polynomial of x with respect to x. We will also need to know how to differentiate a function which is the sum of two simple functions of the form kx^n. It is easy to see that the derivative of a sum of functions is the sum of derivatives. To prove this we go back to the definition. Suppose we want the derivative of $f(x) + g(x)$ where f and g are any functions.

$$\frac{d(f+g)}{dx} = lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - f(x) - g(x)}{\Delta x}$$

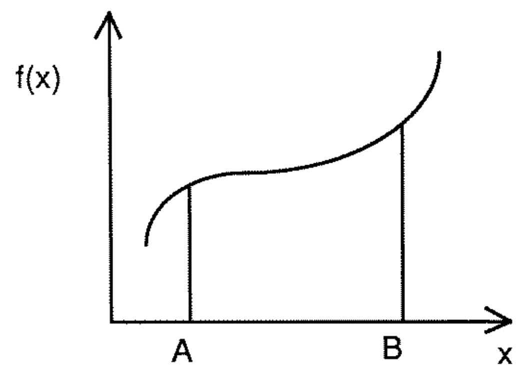
$$= lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right]$$

$$= \frac{df}{dx} + \frac{dg}{dx}.$$

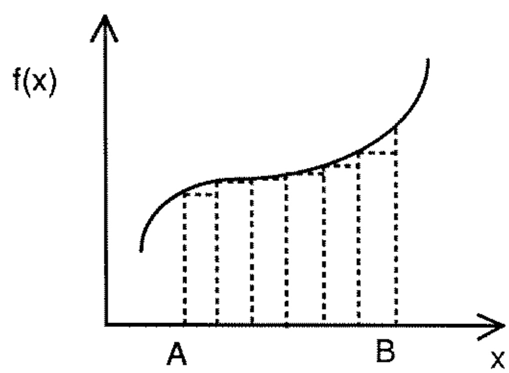
This is all we will need to know about derivatives for some time.

Introduction to Integrals

The integral of a function is related to the area under a curve. Because of this, integrals are really easier to understand than derivatives. Suppose we have a function $f(x)$ as shown below:



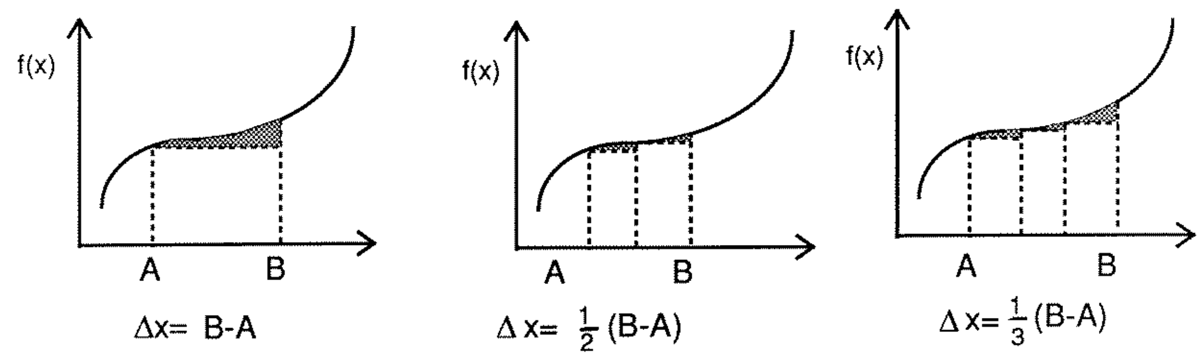
We want to know the area under the curve $f(x)$ between the points $x = A$ and $x = B$. (This may seem to be an absurd thing to be interested in, but it is very useful!) We could get an approximate answer by dividing the interval between A and B into segments which have a length Δx



Then the area under the curve is roughly the sum of the areas of a bunch of rectangles where each rectangle has a base of length Δx and a height given by the value of $f(x)$ at the left hand edge of the rectangle. Calling the sum of these areas I we have

$$I = f(A)\Delta x + f(A + \Delta x)\Delta x + f(A + 2\Delta x)\Delta x + \cdots + f(B - \Delta x)\Delta x.$$

Now I will depend on the size of Δx because we are making an error in calculating the area under $f(x)$ - specifically we are leaving out the little pieces not contained in our rectangles. For example the errors being made for three different values of Δx are shown in the shaded areas below:

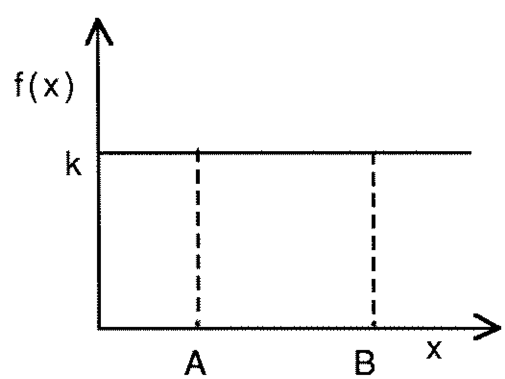


If you think about it for a moment you will see that the smaller we make Δx, the smaller will be the error in our approximation to the area. Of course, we pay a price for getting a more accurate answer in that the smaller we make Δx the larger the number of terms we have to calculate for I. We can now define an integral. It is the value of I when we let Δx go to zero. We write it as

$$\int_A^B f(x)dx$$

and you can think of $f(x)dx$ as the area of a rectangle whose base is dx, infinitesimally small. The integral sign, $\int$, can be thought of as a summation sign, $\sum$, where you have to sum an infinite number of terms. It should be stressed that the integral is just exactly the area under the curve.

To clarify things we'll again consider the simple functions. If $f(x) = k$, a constant,



then

$$\int_A^B f(x)dx = f(A)\Delta x + f(A+\Delta x)\Delta x + \cdots f(B-\Delta x)\Delta x$$

with Δx "going to zero." But here

$$f(A) = f(A+\Delta x) = f(anything) = k$$

so that

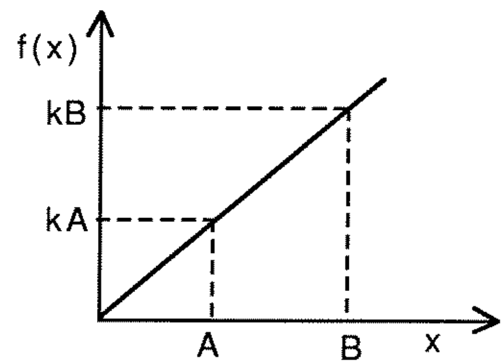
$$\int_A^B f(x)dx = k(\Delta x + \Delta x + \Delta x + \cdots$$

$$= k \cdot (\textit{ the distance between } A \textit{ and } B)$$

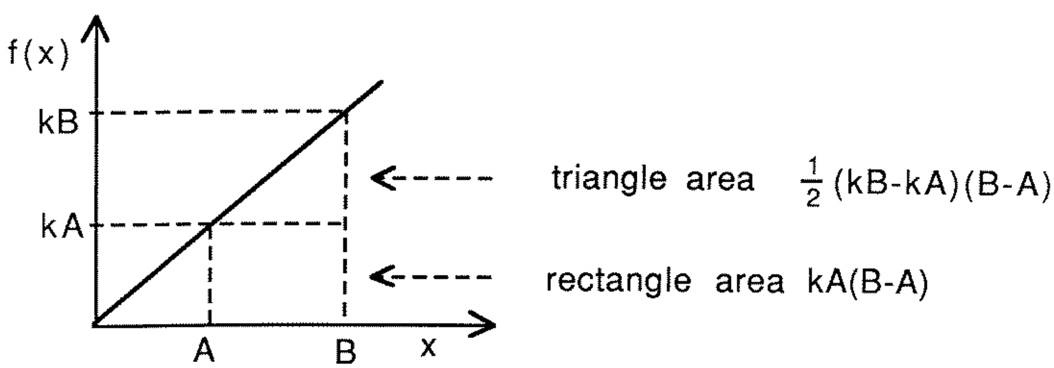
$$= k(B - A),$$

just the base times the height.

For the second example we consider the function $f(x) = kx$:



It is pretty clear that the area we are interested in can be broken up into the area of a rectangle whose height is kA and a triangle whose height is $(kB - kA)$, as shown below.



Then

$$\int_A^B f(x)dx = kA(B - A) + \frac{1}{2}(kB - kA)(B - A)$$

$$= (kA + \frac{1}{2}kB - \frac{1}{2}kA)(B - A)$$

$$= (\frac{1}{2}kA + \frac{1}{2}kB)(B - A) = \frac{1}{2}k(B^2 - A^2).$$

(This is just the difference in area between the triangle with base B and the one with base A.) In this case it is not so simple to arrive at this answer by breaking the desired area into rectangles but we have the answer anyway.

Now its pretty dangerous to try to get a general formula for the integral of a function from these two simple results. Dangerous or not we will do it. We have

$f(x)$	$\int_A^B f(x)dx$
k	$k(B-A)$
kx	$\frac{1}{2}k(B^2-A^2)$

and if you test it (and you should) you will see that

$$\int_A^B (kx^n)dx = \frac{1}{n+1}kB^{n+1} - \frac{1}{n+1}kA^{n+1} \qquad (\textit{ for } n \textit{ any positive integer})$$

again gives back the results we had for $n = 0$ and $n = 1$ and, in fact, it is correct for any number except $n = -1$.

Such integrals are called definite integrals because we are calculating the area between two definite points, called the *limits of integration.* A more useful quantity is the indefinite integral which we now define. Suppose we do not specify the value of B but rather let B be any value of x. Furthermore, suppose we assume that we start from $A = 0$. Then, for the simplest function we would have

$$\int_0^x kdx = kx.$$

or, omitting the upper limit of integration,

$$\int_0 kdx = kx.$$

Now if we want to start this integral at any other value of A we will have to subtract the area from zero to the desired value of A. If we leave the limits undefined then the "correction" for where the integral starts is called the *constant of integration.* The integral is then called an *indefinite integral,* with no upper or lower limits. We indicate all of this by writing

$$\int kdx = kx + \textit{ constant of integration}$$

$$\int kxdx = \frac{1}{2}kx^2 + \textit{ constant of integration}$$

and, in general,

$$\int (kx^n)dx = \frac{1}{n+1}kx^{n+1} + \textit{ constant of integration.}$$

If this is the first time you have seen an integral, I would be surprised if you felt comfortable with this constant of integration business. Just relax. This is something which will become crystal clear when we use it in the next chapter.

Relation Between Integration and Differentiation

We end this chapter by demonstrating a very important relation between integrals and derivatives. Suppose we have the function

$$f(x) = kx$$

and we differentiate with respect to x:

$$\frac{df}{dx} = k.$$

Suppose we now integrate the result

$$\int k dx = \int kx^0 dx \qquad (since\ x^0 = 1)$$

$$= \frac{1}{0+1} kx^{0+1} + \ constant\ of\ integration$$

$$= kx + c \qquad (Here\ we\ call\ the\ constant\ c.)$$

Thus if we start with this simple function, differentiate with respect to x and then integrate we come back to the same function - except for the pesky constant of integration.

Exercise: Prove that the same thing happens if you start with $f(x) = kx^2$.

We now prove that the same thing happens for the general function $f(x) = kx^n$. According to our rule

$$\frac{df}{dx} = nkx^{n-1}.$$

To use our handy integral formula we note that this new function is a constant, (nk), times x raised to the $n-1$ power. Our formula then says the answer is

$$\frac{1}{the\ power\ +1}\ times\ (the\ constant)\ times\ x^{power+1}.$$

The power was $n-1$ so that the result is

$$\int nkx^{n-1}dx = \frac{1}{(n-1)+1} \cdot (nk) \cdot x^{(n-1)+1} + c$$

$$= \frac{1}{n} \cdot (nk) \cdot x^n + c = kx^n + c$$

so that again- except for the constant of integration c - we get back to where we started by differentiating and then integrating. We can write this as

$$\int \frac{df}{dx} dx = f(x) + c.$$

Exercise: Just to prove to yourself that you can manipulate these new quantities, integrals and derivatives, show that starting with $f(x) = kx^n$ and integrating, and then differentiating the result with respect to x, you get back the original function.

We have seen that integration is the "inverse" procedure of differentiation. The reason this inverse operation contains the "constant of integration" can be seen from the fact that

$$f_1(x) = kx^n$$
$$f_2(x) = kx^n + 5$$

and

$$f_3(x) = kx^n + c \quad (with\ c\ any\ constant)$$

all have the same derivatives:

$$\frac{df_1}{dx} = \frac{df_2}{dx} = \frac{df_3}{dx} = nkx^{n-1}.$$

Since integrating is supposed to bring us back to the original function we must have the freedom to choose an additive constant to make up for the constant that gets lost in the differentiation.

Summary

We defined the derivative of a function $f(x)$ by the equation

$$\frac{df}{dx} = limit_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

We "derived" the formula for the derivative of a polynomial

$$\frac{d(kx^n)}{dx} = nkx^{n-1}.$$

A definite integral was defined as the area under a curve between prescribed points called the *limits of integration*. If the function is a polynomial, then

$$\int_A^B kx^n dx = \frac{1}{n+1} k(B^{n+1} - A^{n+1}) \qquad (for\ n \neq -1)$$

If the limits of integration are not specified, the integral is called an *indefinite integral* and, for polynomials,

$$\int kx^n dx = \frac{1}{n+1} kx^{n+1} + c$$

Integration and differentiation were shown to be inverse operations. In other words, up to a constant, the integral of the derivative of a function is the function itself. Thus

$$\int \frac{df}{dx} dx = f(x) + c.$$

PROBLEMS - CHAPTER I

1. Show that the general formula for the derivative of a polynomial agrees with the definition in terms of limits for the specific function $f(x) = kx^3$.

2. Use the general integration formula to find the area under $f(x) = 3x^3$ between $x = 2$ and $x = 3$.

3. Show that starting with $f(x) = 3x^3$ you get back the same function by integrating and then differentiating.

4. Find the constant of integration if you are told that the value of the integral at $x = 2$ is 10 for the function $f(x) = 3x^2$.

SOLUTIONS - CHAPTER I

1. The general formula gives

$$\frac{d(kx^3)}{dx} = 3kx^2.$$

From the definition

$$\frac{df}{dx} = limit_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

$$= lim_{\Delta x \to 0} \frac{k(x+\Delta x)^3 - kx^3}{\Delta x}$$

$$= lim_{\Delta x \to 0} \frac{k\left(x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right) - kx^3}{\Delta x}$$

$$= lim_{\Delta x \to 0} k\left(3x^2 + 3x\Delta x + (\Delta x)^2\right).$$

The last two terms go to zero as $\Delta x \to 0$ while the first is independent of Δx. Thus

$$\frac{df}{dx} = 3kx^2.$$

2. $\int_2^3 (3x^3)dx = \frac{1}{3+1} \cdot 3 \cdot [3^4 - 2^4] = \frac{3}{4}[81 - 16] = 48\frac{3}{4}$

3. $f(x) = 3x^3$

$$\int 3x^3 dx = \frac{3}{4}x^4 + c$$

$$\frac{d[\frac{3}{4}x^4 + c]}{dx} = \frac{d[\frac{3}{4}x^4]}{dx} + \frac{dc}{dx}$$

$$= 3x^3 + 0.$$

We have used the fact that the derivative of a sum of functions is the sum of their individual derivatives.

4. For $f(x) = 3x^2$

$$I = \int 3x^2 dx = x^3 + \ const$$

At $x = 2$, $I = (2)^3 + c = 8 + c$. We are given that I at $x = 2$ is 10. Therefore $c = 2$.

EXERCISES -CHAPTER I

1. Find the derivative of

   a. $4x^2$ *b.* $5x^5$ *c.* $12x^{\frac{1}{2}}$

2. Use the definition of the derivative in terms of Δx to show that

$$\frac{d[ax^2 + bx]}{dx} = 2ax + b$$

   This shows that "the derivative of the sum of two functions is the sum of the derivatives".

3. Start with the answers to exercise 1 and show that integrating these functions gives back the original functions, except for the constant of integration.

4. Find the derivatives of

   a. $4x^2 + 5x + 11$

   b. $x + \frac{1}{x}$

5. Evaluate the following definite integrals

   a. $\int_0^3 2x^2 dx$

   b. $\int_0^1 [x^3 + x] dx$

   c. $\int_1^3 x^{-2} dx$

   d. $\int_2^3 \frac{d[2x^2]}{dx} dx$

ANSWERS - CHAPTER I

1. a. 8x b. $25x^4$ c. $6x^{-\frac{1}{2}}$

4. a. $8x + 5$ b. $1 - \frac{1}{x^2}$

5. a. 18 b. $\frac{3}{4}$ c. $\frac{2}{3}$ d. 10